

THE t -ANALOGS OF STRING FUNCTIONS FOR $A_1^{(1)}$ AND HECKE INDEFINITE MODULAR FORMS.

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ABSTRACT. We study generating functions for Lusztig's t -analog of weight multiplicities associated to integrable highest weight representations of the simplest affine Lie algebra $A_1^{(1)}$. At $t = 1$, these reduce to the *string functions* of $A_1^{(1)}$, which were shown by Kac and Peterson to be related to certain Hecke indefinite modular forms. Using their methods, we obtain a description of the general t -string function; we show that its values can be realized as radial averages of a certain extension of the Hecke indefinite modular form.

1. INTRODUCTION

1.1. Let \mathfrak{g} be an affine Kac-Moody algebra. Let δ denote its null root. Let Λ be a dominant integral weight of \mathfrak{g} , and $L(\Lambda)$ the corresponding irreducible highest weight representation. A weight λ of $L(\Lambda)$ is *maximal* if $\lambda + \delta$ is not a weight of this module. For a maximal dominant weight λ of $L(\Lambda)$, the *string function* $c_\lambda^\Lambda(\tau)$ is (up to multiplication by a power of $q = e^{2\pi i\tau}$) the generating function of weight multiplicities along the δ -string $\{\lambda - k\delta : k \geq 0\}$. String functions are known to be modular forms of weight $-1/2$ for certain congruence subgroups of $SL_2(\mathbb{Z})$ [1].

We now consider the simplest affine algebra $\mathfrak{g} = A_1^{(1)}$. This is the only case for which an explicit description of all string functions is known.

Theorem 1. ([2]) *Let $\mathfrak{g} = A_1^{(1)}$. Let Λ be a dominant integral weight of \mathfrak{g} , and λ be a maximal dominant weight of $L(\Lambda)$. Then*

$$c_\lambda^\Lambda(\tau) = \theta_L(\tau) \eta(\tau)^{-3}.$$

Here $\theta_L(\tau)$ is a Hecke indefinite modular form and $\eta(\tau)$ is the Dedekind eta function.

We explain these notions further in the next subsection. In this paper, we consider Lusztig's t -analog of weight multiplicities (Kostka-Foulkes polynomials) and the corresponding t -string functions (see section 2.1). The level 1 t -string functions are explicitly known for all simply-laced untwisted affines and for the twisted affines [3, 4]. Our present goal is to obtain a description of all t -string functions for $\mathfrak{g} = A_1^{(1)}$, thereby generalizing theorem 1.

1.2. In order to state our main theorem, we recall some background from [2]. Let $\mathfrak{g} = A_1^{(1)}$. Fix a dominant integral weight Λ of \mathfrak{g} of level $m \geq 1$, and let λ be a maximal dominant weight of $L(\Lambda)$. Let N denote the quadratic form defined on \mathbb{R}^2 by:

$$N(x, y) := 2(m+2)x^2 - 2my^2 \quad (x, y \in \mathbb{R})$$

and let $(\cdot|\cdot)$ denote the corresponding symmetric bilinear form. Let $M := \mathbb{Z}^2$ and let M^* denote the lattice dual to M with respect to this form.

Let $O(N)$ denote the group of invertible linear operators on \mathbb{R}^2 preserving N , and $SO_0(N)$ be the connected component of $O(N)$ containing the identity. We then have the groups $G := \{g \in SO_0(N) : gM = M\}$ and $G_0 := \{g \in G : g \text{ fixes } M^*/M \text{ pointwise}\}$. The set $U^+ := \{(x, y) \in \mathbb{R}^2 : N(x, y) > 0\}$ is preserved under the action of $O(N)$ on \mathbb{R}^2 . We let $A := \frac{(\Lambda + \rho, \check{\alpha}_1)}{2(m+2)}$ and $B := \frac{(\lambda, \check{\alpha}_1)}{2m}$ where $\check{\alpha}_1$ is the coroot corresponding to the underlying finite type diagram (\mathfrak{sl}_2 in this case), and ρ is the Weyl vector. Then, $(A, B) \in M^*$, and we set $L := (A, B) + M$. The Hecke indefinite modular form that occurs in theorem 1 is the following sum:

$$\theta_L(\tau) := \sum_{\substack{(x,y) \in L \cap U^+ \\ (x,y) \bmod G_0}} \text{sign}(x, y) e^{\pi i \tau N(x, y)},$$

where $\text{sign}(x, y) = 1$ for $x \geq 0$ and -1 for $x < 0$. This is an absolutely convergent sum for τ in the upper half plane \mathbb{H} , and defines a cusp form of weight 1. We set $\mathbb{D} := \{\omega \in \mathbb{C} : |\omega| < 1\}$, and $\overline{\mathbb{D}}$ its closure in the metric topology.

We now consider the group $\tilde{G} := \langle \zeta \rangle \ltimes G$ where $\zeta \in O(N)$ is defined by $\zeta(x, y) := (-x, y)$. We have

$$\tilde{\mathbf{F}} := \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \text{ or } 0 > y > x\}$$

is a fundamental domain for the action of \tilde{G} on U^+ . Given $(x, y) \in \mathbb{R}^2$, we let (x^\dagger, y^\dagger) denote the unique element of $\tilde{\mathbf{F}}$ which is in the \tilde{G} -orbit of (x, y) . We now extend $\theta_L(\tau)$ to a function $\vartheta_L(\omega; \tau)$ on $\overline{\mathbb{D}} \setminus \{0\} \times \mathbb{H}$ as follows:

$$\vartheta_L(te^{2\pi i u}; \tau) := \sum_{\substack{(x,y) \in L \cap U^+ \\ (x,y) \bmod G_0}} \text{sign}(x, y) e^{\pi i \tau N(x, y)} t^{2(y^\dagger - B)} e^{2\pi i u((m+2)x^\dagger - my^\dagger - \frac{1}{2})} \quad (1.1)$$

where $0 < t \leq 1$ and $-1/2 \leq u < 1/2$. This turns out to be a well-defined function, continuous in $\omega = te^{2\pi i u}$ and holomorphic in τ . It can be viewed as a (specialization of a) kind of theta function associated to the indefinite lattice L .

We also extend $\eta(\tau)^{-3}$ to the function $\eta^{(-3)}(\omega; \tau) : \overline{\mathbb{D}} \times \mathbb{H} \rightarrow \mathbb{C}$ defined by

$$\eta^{(-3)}(\omega; \tau) := e^{-\pi i \tau / 4} \prod_{n=1}^{\infty} \frac{1}{(1 - e^{2\pi i n \tau})(1 - \omega e^{2\pi i n \tau})(1 - \overline{\omega} e^{2\pi i n \tau})}.$$

Clearly, $\vartheta_L(1; \tau) = \theta_L(\tau)$ and $\eta^{(-3)}(1; \tau) = \eta(\tau)^{-3}$.

Finally, recall that the Poisson kernel $P(\omega) : \mathbb{D} \rightarrow \mathbb{C}$ is the positive harmonic function defined by:

$$P(\omega) = \frac{1 - |\omega|^2}{(1 - \omega)(1 - \bar{\omega})}.$$

We set $P_t(u) := P(te^{2\pi i u})$.

The following, which is our main theorem, states that the values of the t -string function $c_\lambda^\Lambda(t, \tau)$ for $0 < t < 1$ are the radial averages of $\vartheta_L(\omega; \tau) \eta^{(-3)}(\omega; \tau)$ with respect to the measure defined by $P_t(u)$.

Theorem 2. *Let $F(\omega; \tau) := \vartheta_L(\omega; \tau) \eta^{(-3)}(\omega; \tau)$. Then*

$$c_\lambda^\Lambda(t, \tau) = \int_0^1 F(te^{2\pi i u}; \tau) P_t(u) du.$$

We recall that the collection $\{P_t(u)\}$ is an approximate identity on the unit circle; as $t \rightarrow 1$, the measure defined by $P_t(u)$ approaches the Dirac delta measure supported on the single point 1. In this limit, we have

$$\int_0^1 F(te^{2\pi i u}; \tau) P_t(u) du \rightarrow \vartheta_L(1; \tau) \eta^{(-3)}(1; \tau) = \theta_L(\tau) \eta(\tau)^{-3},$$

giving us back theorem 1.

The proof of theorem 2 occupies the rest of the paper. Our proof closely follows that of theorem 1 by Kac and Peterson [2].

2.

2.1. We assume throughout that $\mathfrak{g} = A_1^{(1)}$. Let \mathfrak{h} denote the Cartan subalgebra of \mathfrak{g} , $K \in \mathfrak{h}$ the canonical central element, Δ_+ the set of positive roots and δ the null root. Let Q, P denote the root and weight lattices of \mathfrak{g} . The standard basis [1, chapter 6] of \mathfrak{h}^* is $\{\alpha_1, \delta, \Lambda_0\}$, where α_1 is the simple root corresponding to the underlying \mathfrak{sl}_2 diagram, and Λ_0 is a fundamental weight corresponding to the extended node. Given $\lambda \in \mathfrak{h}^*$ of level $m = \langle \lambda, K \rangle$, we have $\lambda = \mathbf{b}(\lambda)\alpha_1 + \mathbf{d}(\lambda)\delta + m\Lambda_0$ for unique scalars $\mathbf{b}(\lambda)$ and $\mathbf{d}(\lambda)$.

The t -Kostant partition function is given by:

$$K(\beta; t) := [e(-\beta)] \prod_{\alpha \in \Delta_+} \frac{1}{(1 - te(-\alpha))^{m_\alpha}},$$

i.e., the coefficient of $e(-\beta)$ in the product. For $\mathfrak{g} = A_1^{(1)}$, $m_\alpha = 1$ for all $\alpha \in \Delta_+$. Lusztig's t -analog of weight multiplicity or (affine) Kostka-Foulkes polynomial $m_\mu^\lambda(t)$ is defined by

$$m_\mu^\lambda(t) := \sum_{w \in W} \epsilon(w) K(w(\lambda + \rho) - (\mu + \rho); t)$$

where λ, μ are dominant integral weights, W is the Weyl group, and ϵ is its sign character.

For $\beta \in Q$, define the function $K'(\beta; t)$ as follows:

$$K'(\beta; t) := K(\beta; t) + t K(r_1 \cdot \beta; t).$$

We have the following simple observation: $K(\beta; t) = 0$ iff $\beta \notin Q_+$, and $K'(\beta; t) = 0$ iff $\beta \notin Q_+ \cup r_1 \cdot Q_+$ iff $d(\beta) \leq -1$.

We consider the corresponding generating functions for values along δ -strings, defined by (for $\beta \in Q$, $\lambda, \mu \in P^+$):

$$\mathbf{K}_\beta := \sum_{n \geq 0} K(\beta + n\delta; t) q^n \quad (2.1)$$

$$\mathbf{K}'_\beta := \sum_{n \geq 0} K'(\beta + n\delta; t) q^n \quad (2.2)$$

$$a_\mu^\lambda(t, q) := \sum_{k \geq 0} m_{\mu - k\delta}^\lambda(t) q^k \quad (2.3)$$

where $q := e(-\delta)$. We will also think of these as functions of $\tau \in \mathbb{H}$ by setting $q := e^{2\pi i \tau}$. Now, let

$$\Gamma_t := \prod_{\alpha \in \Delta_+} \frac{1}{1 - te(-\alpha)} = \sum_{\beta \in Q_+} K(\beta; t) e(-\beta)$$

$$\xi_t := \frac{1}{\prod_{n \geq 1} (1 - tq^n)(1 - tq^n e(-\alpha_1))(1 - tq^n e(\alpha_1))}$$

Then, we observe that $\Gamma_t = \frac{\xi_t}{1 - te(-\alpha_1)}$.

2.2. We recall the *constant term* map $\text{ct}(\cdot)$, defined on formal sums $\sum_{\alpha \in Q} c_\alpha e(\alpha)$ by $\text{ct}(\sum_{\alpha \in Q} c_\alpha e(\alpha)) := \sum_{n \in \mathbb{Z}} c_{n\delta} e(n\delta)$. Let $\mathcal{L} := \{\beta \in Q : d(\beta) \leq 0\}$.

Lemma 1. *If $\beta \in \mathcal{L}$, then*

$$\mathbf{K}'_\beta = \text{ct}(e(\beta) \xi_t P_t) \quad (2.4)$$

where $P_t := \sum_{n \in \mathbb{Z}} t^{|n|} e(n\alpha_1)$ is the (formal) Poisson kernel of the unit disc.

Proof. Let $\beta \in \mathcal{L}$. Observe that in this case $r_1 \cdot \beta \in \mathcal{L}$, and the sums on the right hand sides of equations (2.1) and (2.2) can be replaced by $\sum_{n \in \mathbb{Z}}$. It then follows from definitions that (i) $\mathbf{K}'_\beta = \mathbf{K}_\beta + t \mathbf{K}_{r_1 \cdot \beta}$, (ii) $\mathbf{K}_\beta = \text{ct}(\Gamma_t e(\beta))$ and (iii) $\mathbf{K}_{r_1 \cdot \beta} = \text{ct}(\Gamma_t e(r_1 \cdot \beta))$.

For $\xi = \sum_{\lambda \in \mathfrak{h}^*} c_\lambda(t) e(\lambda)$, define $\bar{\xi} := \sum_{\lambda \in \mathfrak{h}^*} c_\lambda(t) e(r_1(\lambda))$. Note that $\text{ct}(\xi) = \text{ct}(\bar{\xi})$. For $\xi = \Gamma_t e(r_1 \cdot \beta)$, we have $\bar{\xi} = \bar{\Gamma}_t e(\beta + \alpha_1)$. Further, it is easy to see that $\Gamma_t + t e(\alpha_1) \bar{\Gamma}_t = P_t \xi_t$. Putting all these facts together completes the proof. \square

Let W denote the Weyl group of \mathfrak{g} ; it can be written as $W = T \rtimes \overset{\circ}{W}$ where T is the group of translations by elements of the finite root lattice $\overset{\circ}{Q} = \mathbb{Z}\alpha_1$, and $\overset{\circ}{W} = \{1, r_1\}$ is the Weyl group of the underlying \mathfrak{sl}_2 . The extended affine Weyl group $\widehat{W} := \widehat{T} \rtimes \overset{\circ}{W}$ where \widehat{T} is the set of translations

by elements of the finite weight lattice. Let t_α denote the translation by the element α of the finite weight lattice. Define $\tau := t_{\alpha_1/2}$, and $\sigma := \tau r_1$. Then $T = \{\tau^{2n} : n \in \mathbb{Z}\}$, and the element σ permutes the simple roots of \mathfrak{g} , $\sigma(\alpha_0) = \alpha_1$ and $\sigma(\alpha_1) = \alpha_0$, and fixes the Weyl vector ρ . We also have the following formula for the action of τ [1]:

$$\tau(\lambda) = \lambda + \frac{1}{2}(\langle \lambda, \delta \rangle \alpha_1 - \langle \lambda, \alpha_1 \rangle \delta - \langle \lambda, \delta \rangle \frac{\delta}{2}). \quad (2.5)$$

We note that $\langle \lambda, \delta \rangle$ is the level of λ . Define a function $I : Q \times \mathbb{Z} \rightarrow \{0, \pm 1\}$ by

$$I(\beta, j) := \begin{cases} 1 & \text{if } \mathbf{b}(\beta) \geq 0, j \geq 0. \\ -1 & \text{if } \mathbf{b}(\beta) < 0, j < 0. \\ 0 & \text{otherwise.} \end{cases} \quad (2.6)$$

Lemma 2. *Let $\beta \in Q$. Then*

$$\mathbf{K}_\beta = \sum_{j \in \mathbb{Z}} (-1)^j I(\beta, j) t^j \mathbf{K}'_{\tau^j \cdot \beta}$$

Proof. Since σ interchanges the simple roots α_0, α_1 and fixes ρ , it is clear that $K(\beta; t) = K(\sigma\beta; t) = K(\sigma \cdot \beta; t)$ for all $\beta \in Q$. Now, this implies that $K(\beta; t) = K'(\beta; t) - t K(r_1 \cdot \beta; t) = K'(\beta; t) - t K(\tau \cdot \beta; t)$. Iterating the last expression gives

$$K(\beta; t) = \sum_{j \geq 0} (-1)^j t^j K'(\tau^j \cdot \beta; t). \quad (2.7)$$

Similary, replacing β by $\sigma\beta$, one obtains the relations $K(\beta; t) = t^{-1} K'(\beta; t) - t^{-1} K(\tau^{-1} \cdot \beta; t)$ and hence

$$K(\beta; t) = - \sum_{j < 0} (-1)^j t^j K'(\tau^j \cdot \beta; t). \quad (2.8)$$

The sums in equations (2.7) and (2.8) are in fact finite (as can be seen from equation (2.10) below) and either expression can be used for a given $\beta \in Q$. But choosing the expression (2.7) (resp. (2.8)) when $\mathbf{b}(\beta) \geq 0$ (resp. $\mathbf{b}(\beta) < 0$), we obtain

$$K(\beta; t) = \sum_{j \in \mathbb{Z}} (-1)^j I(\beta, j) t^j K'(\tau^j \cdot \beta; t). \quad (2.9)$$

To complete the proof, it only remains to replace β by $\beta + n\delta$ ($n \geq 0$) in (2.9) and observe that (i) $I(\beta + n\delta, j) = I(\beta, j)$ and (ii) $\tau^j \cdot (\beta + n\delta) = (\tau^j \cdot \beta) + n\delta$. \square

Lemma 3. *Let $\beta \in Q$.*

- (1) $\{j \in \mathbb{Z} : I(\beta, j) \neq 0\} \subset \{j \in \mathbb{Z} : \mathbf{d}(\tau^j \cdot \beta) \leq \mathbf{d}(\beta)\}$.
- (2) *If $I(\beta, j) \neq 0$ and $\beta \in \mathcal{L}$, then $\tau^j \cdot \beta \in \mathcal{L}$.*

Proof. The second assertion clearly follows from the first. To prove (1), we use equation (2.5) to obtain:

$$\tau^j \cdot \beta - \beta = \tau^j(\beta + \rho) - (\beta + \rho) = j\alpha_1 - \left(j \mathbf{b}(\beta) + \frac{j(j+1)}{2} \right) \delta. \quad (2.10)$$

Thus $\mathbf{d}(\tau^j \cdot \beta - \beta) = -j \mathbf{b}(\beta) - \frac{j(j+1)}{2}$. It is clear from equation (2.6) that this is non-positive for all pairs (β, j) for which $I(\beta, j) \neq 0$. \square

2.3. We will henceforth fix Λ , a dominant integral weight of \mathfrak{g} , and λ a maximal dominant weight of $L(\Lambda)$. We may assume without loss of generality that $\Lambda - \lambda \in \mathbb{Z}\alpha_1$. For $w \in W$, define

$$s(w) := w(\Lambda + \rho) - (\lambda + \rho) = w \cdot \Lambda - \lambda \in Q.$$

Lemma 4. $s(w) \in \mathcal{L}$ for all $w \in W$.

Proof. We have $\mathbf{d}(\beta) = \langle \beta, \Lambda_0 \rangle$ for all $\beta \in Q$. Thus $\mathbf{d}(s(w)) = \langle w(\Lambda + \rho) - (\lambda + \rho), \Lambda_0 \rangle + \langle \Lambda - \lambda, \Lambda_0 \rangle$. The second term is zero since $\Lambda - \lambda \in \mathbb{Z}\alpha_1$, while the first term equals $\langle \Lambda + \rho, w^{-1}\Lambda_0 - \Lambda_0 \rangle$ which is non-positive since Λ_0 is a dominant weight. \square

Lemma 5. $a_\lambda^\Lambda(t, q) = \sum_{w \in W} (-1)^{\ell(w)} \mathbf{K}_{s(w)}$

Proof. This follows from the definitions. \square

By lemma 2, we get

$$a_\lambda^\Lambda(t, q) = \sum_{w \in W} \sum_{j \in \mathbb{Z}} (-1)^{\ell(w)+j} I(s(w), j) t^j \mathbf{K}'_{\tau^j \cdot s(w)}.$$

By lemmas 3 and 4, it is clear that $\tau^j \cdot s(w) \in \mathcal{L}$ for all pairs (w, j) for which $I(s(w), j) \neq 0$. Thus by lemma 1, $\mathbf{K}'_{\tau^j \cdot s(w)} = \text{ct}(P_t \xi_t e(\tau^j \cdot s(w)))$.

Now, define a function $\bar{e} : W \times \mathbb{Z} \rightarrow \{0, \pm 1\}$ by $\bar{e}(w, j) := (-1)^{\ell(w)+j} I(s(w), j)$, and let:

$$\bar{\mathcal{H}} := \sum_{(w, j) \in W \times \mathbb{Z}} \bar{e}(w, j) t^j e(\tau^j \cdot s(w)).$$

Then $a_\lambda^\Lambda(t, q) = \text{ct}(P_t \xi_t \bar{\mathcal{H}})$.

3.

3.1. Let $U := \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_1$ and $M := \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_1$. We identify U with \mathbb{R}^2 and M with \mathbb{Z}^2 . Define a quadratic form N on U by:

$$N(x, y) := 2(m+2)x^2 - 2my^2, x, y \in \mathbb{R}.$$

We observe that $N(\nu)$ is a non-zero even integer for $\nu \in M \setminus \{0\}$.

The dual lattice $M^* = \frac{1}{2(m+2)}\mathbb{Z} \oplus \frac{1}{2m}\mathbb{Z}$. Given elements $\mu_1, \mu_2 \in P$ of levels $m+2$ and m respectively, observe that $\left(\frac{\mathbf{b}(\mu_1)}{m+2}, \frac{\mathbf{b}(\mu_2)}{m} \right) \in M^*$, since $\mathbf{b}(\mu_i) = \frac{\langle \mu_i, \alpha_1 \rangle}{2} \in \frac{1}{2}\mathbb{Z}$.

Lemma 6. For $(w, j) \in W \times \mathbb{Z}$, we have

$$\tau^j \cdot s(w) = \left((m+2)x - my - \frac{1}{2} \right) \alpha_1 - \frac{1}{2} N(x, y) \delta + (s_\Lambda(\lambda) + \frac{1}{8}) \delta$$

where $x := \frac{j}{2} + \frac{b(w(\Lambda+\rho))}{m+2}$ and $y := \frac{j}{2} + \frac{b(\lambda)}{m}$.

Proof. This is an easy calculation. The coefficient of δ was computed in [2, (5.13)] for $w \in T$. \square

Corollary 1. $c_\lambda^\Lambda(t, q) := q^{s_\Lambda(\lambda)} a_\lambda^\Lambda(t, q) = \text{ct}(P_t(q^{-\frac{1}{8}} \xi_t) \vartheta)$ where

$$\vartheta := \sum_{(w, j) \in W \times \mathbb{Z}} \bar{e}(w, j) t^j q^{\frac{1}{2} N(x, y)} z^{(m+2)x - my - 1/2}. \quad (3.1)$$

Here $z := e(\alpha_1)$, and $x, y \in M^*$ are the functions of (w, j) defined in lemma 6.

We remark that $q^{-\frac{1}{8}} \xi_t$ reduces to $\eta(q)^{-3}$ at $t = 1$, where $\eta(q)$ is the Dedekind eta function.

We now turn to the map $\phi : W \times \mathbb{Z} \rightarrow M^*$ given by $(w, j) \mapsto (x, y)$ where

$$x := \frac{j}{2} + \frac{b(w(\Lambda + \rho))}{m+2} \text{ and } y := \frac{j}{2} + \frac{b(\lambda)}{m}$$

as in lemma 6. Let $\phi(e, 0) = (A, B)$ where e is the identity element of W . Thus $A = \frac{b(\Lambda+\rho)}{m+2}$ and $B = \frac{b(\lambda)}{m}$. As in [2], we also assume (without loss of generality) that $0 \leq B \leq A < \frac{1}{2}$.

Lemma 7. (1) ϕ is injective.

(2) $\text{Im } \phi = \bigsqcup_{i=1}^4 L_i$, where

$$L_1 := (A, B) + M \quad (3.2)$$

$$L_2 := (A + \frac{1}{2}, B + \frac{1}{2}) + M \quad (3.3)$$

$$L_3 := (-A, B) + M \quad (3.4)$$

$$L_4 := (-A + \frac{1}{2}, B + \frac{1}{2}) + M \quad (3.5)$$

i.e., a union of translates of M .

Proof. Using lemma 6, it is clear that $\phi(w_1, j_1) = \phi(w_2, j_2)$ implies $j_1 = j_2$ and $\tau^{j_1} \cdot s(w_1) = \tau^{j_2} \cdot s(w_2)$. In turn, this means $s(w_1) = s(w_2)$, and hence $w_1 = w_2$, since $\Lambda + \rho$ is regular dominant. This proves (1).

Next, let $(w, j) \in W \times \mathbb{Z}$. Recall that since $W = T \rtimes \overset{\circ}{W}$, w can be uniquely written as $\tau^{2n} \omega$ for some $n \in \mathbb{Z}, \omega \in \overset{\circ}{W} = \{1, r_1\}$. Now, equation (2.5) implies that

$$x = \frac{j}{2} + n + (\text{sgn } \omega)A, \quad y = \frac{j}{2} + B \quad (3.6)$$

where sgn is the sign character of $\overset{\circ}{W}$. It is now clear that if $W \times \mathbb{Z}$ is written as the disjoint union of the four subsets $S_1 := T \times 2\mathbb{Z}$, $S_2 := T \times (2\mathbb{Z} + 1)$, $S_3 := Tr_1 \times 2\mathbb{Z}$, $S_4 := Tr_1 \times (2\mathbb{Z} + 1)$, then $\phi(S_i) = L_i$ for $1 \leq i \leq 4$. \square

We remark that our assumption on A, B ensures that the L_i are pairwise disjoint. From lemma 7, we see that ϑ has the following equivalent expression:

$$\vartheta = \sum_{(x,y) \in \bigsqcup_{i=1}^4 L_i} \epsilon(x, y) t^{2(y-B)} q^{\frac{1}{2}N(x,y)} z^{(m+2)x-my-1/2} \quad (3.7)$$

where $\epsilon(x, y) := \bar{\epsilon}(\phi^{-1}(x, y))$ for $(x, y) \in \bigsqcup_{i=1}^4 L_i$.

Next, we analyze the set of pairs (x, y) for which $\epsilon(x, y) \neq 0$.

Lemma 8. *For $1 \leq i \leq 4$, we have*

$$\{(x, y) \in L_i : \epsilon(x, y) \neq 0\} = L_i \cap \tilde{\mathbf{F}}$$

where $\tilde{\mathbf{F}} := \{(u, v) \in U : 0 \leq v \leq u \text{ or } 0 > v > u\}$.

Proof. We prove this only for $i = 1$, the rest of the cases being similar. Fix $(x, y) \in L_1$; by lemma 7, we have $(x, y) = \phi(w, j)$ where $w = \tau^{2n}$, $n \in \mathbb{Z}$, $j \in 2\mathbb{Z}$. Now $\epsilon(x, y) \neq 0$ iff $I(s(w), j) \neq 0$ iff either (i) $n, j \geq 0$ or (ii) $n, j < 0$. From equation (3.6) and our assumption that $0 \leq B \leq A < \frac{1}{2}$, it follows that (i) is equivalent to $0 \leq y \leq x$ and (ii) is equivalent to $0 > y > x$. \square

3.2. Let $O(U, N)$ denote the group of invertible linear operators on U preserving the quadratic form N , and let $SO_0(U, N)$ be the connected component of $O(U, N)$ containing the identity. Let $a \in GL(U)$ be defined by

$$a(u, v) := ((m+1)u + mv, (m+2)u + (m+1)v).$$

Let G be the subgroup of $GL(U)$ generated by a , and G_0 be the subgroup of G generated by a^2 . It is known that

$$G = \{g \in SO_0(U, N) : gM = M\}$$

We note that elements of G also leave M^* invariant, and thus G has a natural action on M^*/M . It is known that

$$G_0 = \{g \in G : g \text{ fixes } M^*/M \text{ pointwise}\}.$$

Define $\zeta \in O(U, N)$ by $\zeta(u, v) := (-u, v)$, and let

$$\tilde{G} := \langle \zeta \rangle \ltimes G \text{ and } \tilde{G}_0 := \langle \zeta \rangle \ltimes G_0.$$

We have the following easy properties: (i) ζ^2 is the identity, (ii) $\zeta a \zeta^{-1} = a^{-1}$, (iii) \tilde{G} is an infinite dihedral group. We have the following diagram of inclusions between the four groups. Each inclusion is as an index 2 subgroup.

$$\begin{array}{ccc} G_0 & \longrightarrow & G \\ \downarrow & & \downarrow \\ \tilde{G}_0 & \longrightarrow & \tilde{G} \end{array}$$

Observe that \tilde{G} leaves M and M^* invariant, and hence acts on M^*/M . We now show that the \tilde{G} -orbit of L_1 is $\{L_i : i = 1 \cdots 4\}$.

Lemma 9. (1) *If $g \in G_0$ then $gL_i = L_i$ for $i = 1 \cdots 4$.*
 (2) $L_1 = aL_4 = \zeta L_3 = a\zeta L_2$.

Proof. The first statement follows from the fact that G_0 fixes M^*/M pointwise. To show $L_1 = aL_4$, observe using lemma 6 that $a^{-1}(A, B) = (-A + \frac{1}{2}, B + \frac{1}{2}) + \mathbf{b}(s(e))(1, -1)$. Since $s(e) = \Lambda - \lambda \in Q$, $\mathbf{b}(s(e)) \in \mathbb{Z}$ and we are done. The remaining two equalities are obvious. \square

Let $U^+ := \{(u, v) \in U : N(u, v) > 0\}$.

Lemma 10. (1) U^+ is \tilde{G} -invariant.
 (2) $\tilde{\mathbf{F}}$ is a fundamental domain for the action of \tilde{G} on U^+ .
 (3) $\mathbf{F}_0 := \tilde{\mathbf{F}} \cup a\tilde{\mathbf{F}} \cup \zeta\tilde{\mathbf{F}} \cup a\zeta\tilde{\mathbf{F}}$ is a fundamental domain for the action of G_0 on U^+ .

Proof. (1) is clear since $\tilde{G} \subset O(U, N)$. Now, $\mathbf{F} := \tilde{\mathbf{F}} \cup \zeta\tilde{\mathbf{F}}$ and $\mathbf{F}_0 = \mathbf{F} \cup a\mathbf{F}$ are known to be fundamental domains for the actions of G and G_0 (respectively) on U^+ [2]. It follows that $\tilde{\mathbf{F}}$ is a fundamental domain for the action of \tilde{G} on U^+ . \square

Lemmas 9 and 10 allow us to identify the sets $\bigsqcup_{i=1}^4 L_i \cap \tilde{\mathbf{F}}$ and $L_1 \cap \mathbf{F}_0$. More precisely, define the map $\psi : \bigsqcup_{i=1}^4 L_i \cap \tilde{\mathbf{F}} \rightarrow L_1 \cap \mathbf{F}_0$ by

$$\psi(x, y) := \begin{cases} (x, y) & \text{if } (x, y) \in L_1 \cap \tilde{\mathbf{F}} \\ a\zeta(x, y) & \text{if } (x, y) \in L_2 \cap \tilde{\mathbf{F}} \\ \zeta(x, y) & \text{if } (x, y) \in L_3 \cap \tilde{\mathbf{F}} \\ a(x, y) & \text{if } (x, y) \in L_4 \cap \tilde{\mathbf{F}}. \end{cases} \quad (3.8)$$

By lemmas 9 and 10, it is clear that ψ is well-defined, and is a bijection. In fact, the inverse map ψ^{-1} is easy to describe. Given $(x, y) \in L_1 \cap \mathbf{F}_0$, $\psi^{-1}(x, y)$ is the unique element in the \tilde{G} -orbit of (x, y) which lies in $\tilde{\mathbf{F}}$. We will denote $\psi^{-1}(x, y) =: (x^\dagger, y^\dagger)$.

3.3. We now return to ϑ in equation (3.7) :

$$\vartheta = \sum_{(x, y) \in \bigsqcup_{i=1}^4 L_i} \epsilon(x, y) t^{2(y-B)} q^{\frac{1}{2}N(x, y)} z^{(m+2)x - my - 1/2}$$

where $z := e(\alpha_1)$. Using lemma 8, we can split this into four sums, one over each $L_i \cap \tilde{\mathbf{F}}$. We then perform a change of variables, replacing $(x, y) \in$

$\bigsqcup_{i=1}^4 L_i$ by $\psi(x, y) \in L_1 \cap \mathbf{F}_0$. Since $N(x, y) = N(x^\dagger, y^\dagger)$, the resulting sum becomes:

$$\vartheta = \sum_{(x,y) \in L_1 \cap \mathbf{F}_0} \epsilon(x^\dagger, y^\dagger) t^{2(y^\dagger - B)} q^{\frac{1}{2}N(x,y)} z^{(m+2)x^\dagger - my^\dagger - 1/2}.$$

For $(x, y) \in U^+$, define $\text{sign}(x, y) := 1$ if $x > 0$ and -1 if $x < 0$. We then have:

Lemma 11. *For $(x, y) \in L_1 \cap \mathbf{F}_0$, $\epsilon(x^\dagger, y^\dagger) = \text{sign}(x, y)$.*

Proof. As in the above discussion, we split this into the four cases $(x, y) \in L_1 \cap g\tilde{\mathbf{F}}$ for (i) $g = e$, (ii) $g = a\zeta$, (iii) $g = \zeta$ and (iv) $g = a$. We only consider case (ii), which is representative of the calculation needed for the other cases. For $(x, y) \in L_1 \cap a\zeta\tilde{\mathbf{F}}$, we have $(x^\dagger, y^\dagger) = (a\zeta)^{-1}(x, y) \in L_2 \cap \tilde{\mathbf{F}}$. Let $(x^\dagger, y^\dagger) = \phi(w, j)$ where $w = \tau^{2n}$, $n \in \mathbb{Z}$, $j \in 2\mathbb{Z} + 1$. Now $\epsilon(x^\dagger, y^\dagger) = \bar{\epsilon}(w, j) = -I(s(w), j)$. Now, $\epsilon(x^\dagger, y^\dagger)$ equals -1 if $n, j \geq 0$ and 1 if $n, j < 0$. In other words $\epsilon(x^\dagger, y^\dagger) = -\text{sign}(x^\dagger, y^\dagger) = \text{sign}(x, y)$. The last equality follows from the fact that a leaves sign invariant, while ζ reverses it. \square

Since $\text{sign}(x, y)$ and $N(x, y)$ are constant on G_0 -orbits, we have :

$$\vartheta = t^{-2B} z^{-\frac{1}{2}} \sum_{\substack{(x,y) \in L_1 \cap U^+ \\ (x,y) \bmod G_0}} \text{sign}(x, y) q^{\frac{1}{2}N(x,y)} t^{2y^\dagger} z^{(m+2)x^\dagger - my^\dagger} \quad (3.9)$$

where for $(x, y) \in U^+$, we let (x^\dagger, y^\dagger) denote the unique element in $\tilde{\mathbf{F}} \cap (\tilde{G}$ -orbit of (x, y)).

Finally, we observe that corollary 1 and equation (3.9) imply theorem 2.

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